ON CERTAIN RELATIONS AMONG THE THETA CONSTANTS*

BY

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In the case of the theta functions of two variables $(p = 2)$, the theta constants (i. e., the ten functions with zero arguments) are well known to be expressible in terms of any four $(2^p = 2^2 = 4)$ of them, the most interesting relations being those in which the characteristics of the four form a GÖPEL quadruple. Thus, in the formulæ given by KRAUSE in §11 of his book, *Die Transformation der hyperelliptischen Funktionen erster Ordnung*, we have certain combinations of the constants rationally expressed in terms of ϑ_5, ϑ_0, ϑ_{12}, ϑ_{34} (formulæ (1)), while each constant is expressed irrationally in terms of the same (formulæ (3)).

In the following pages the corresponding results are given for the case of the general theta functions $(p = p)$: viz., formulæ (I), (II) below are a direct generalization of formulæ (1) of KRAUSE just referred to, and (IV) is a generalization of (3), while (III) is a new relation not existing for the case $p = 2$. †

I use the definitions and notations of FROBENIUS (*Das Additionstheorem der Thetafunctionen*, Crelle's Journal, vol. 89, 1880) and start with the formula (loc. cit., p. 201):

$$\sum_{a=0}^{r-1} (P_a)\, \vartheta[A_a]\, (u + v)\, (u - v)\, (a + b)\, (a - b)$$
$$(1)$$
$$= \sum_a (P_a)\, \vartheta[A_a]\, (u + b)\, (u - b)\, (a + v)\, (a - v).$$

In this formula $A_0 = A$, A_1, $\cdots$, A_{r-1} are the $r = 2^p$ characteristics of a GÖPEL system, and $P_a = A A_a$.

In (1) replace u and b by $u + C$ and $b + C$, where C denotes any system of half periods. The formula becomes:

$$(2) \quad \sum (P_a C)\, \vartheta[A_a C]\, (u + v)\, (u - v)\, (a + b)\, (a - b)$$
$$= (C) \sum (P_a C)\, \vartheta[A_a]\, (u + b)\, (u - b)\, (a + v)\, (a - v).$$

Putting $u = v = a = b = 0$ we have

(I) $$\sum (P_a C)\, \vartheta^4[A_a C] = (C) \sum (P_a C)\, \vartheta^4[A_a].$$

In (2) replace u by $u + P_\beta$. We obtain

(3)
$$\sum (P_a C) \binom{P_\beta}{A_a C} \vartheta[A_a P_\beta C]\,(u+v)\,(u-v)\,\vartheta[A_a C]\,(a+b)\,(a-b)$$
$$= (C) \sum (P_a C) \binom{P_\beta}{A_a} \vartheta[A_a P_\beta]\,(u+b)\,(u-b)\,\vartheta[A_a]\,(a+v)\,(a-v).$$

Making the arguments zero we have

(4)
$$\sum (P_a C) \binom{P_\beta}{A_a C} \vartheta^2[A_a P_\beta C]\, \vartheta^2[A_a C]$$
$$= (C) \sum (P_a C) \binom{P_\beta}{A_a} \vartheta^2[A_a P_\beta]\, \vartheta^2[A_a].$$

This formula can be simplified. Suppose the GÖPEL system to be represented in the form

$$AB_i, \quad A_\beta B_i \qquad\qquad (i = 0,\, 1,\, \cdots,\, \tfrac{1}{2}r - 1)$$

where B_0 is the characteristic zero. Then (4) can be written:

$$[1 + (P_\beta, C)] \sum_i (B_i C) \binom{P_\beta}{AB_i C} \vartheta^2[AB_i C]\, \vartheta^2[A_\beta B_i C]$$
$$= (C)\, [1 + (P_\beta, C)] \sum_i (B_i C) \binom{P_\beta}{AB_i} \vartheta^2[AB_i]\, \vartheta^2[A_\beta B_i].$$

In reducing to this form we notice that the B_i are contained among the P_a and hence $(P_\beta, B_i) = 1$.

The formula so obtained vanishes identically if $(P_\beta, C) = -1$. If $(P_\beta, C) = 1$ we have

(II)
$$\sum (B_i C) \binom{P_\beta}{AB_i C} \vartheta^2[AB_i C]\, \vartheta^2[A_\beta B_i C]$$
$$= (C) \sum (B_i C) \binom{P_\beta}{AB_i} \vartheta^2[AB_i]\, \vartheta^2[A_\beta B_i].$$

It is to be observed in this connection that

$$(AB_i)\,(A_\beta B_i) = (A)\,(A_\beta)\,(P_\beta, B_i) = (A)\,(A_\beta),$$

and

$$(AB_i C)\,(A_\beta B_i C) = (A)\,(A_\beta)\,(P_\beta, C) = (A)\,(A_\beta).$$

Consequently if A and A_β are of opposite character (i. e., one odd and the other even), formula (II) vanishes identically.

Again, replacing u, v, a, b by $u + \frac{1}{2}P_\gamma$, $v + \frac{1}{2}P_\gamma$, $a + \frac{1}{2}P_\gamma$, $b + \frac{1}{2}P_\gamma$ in formula (3), we have

$$\sum (P_a C) \begin{pmatrix} P_\beta P_\gamma \\ A_a \ C \end{pmatrix} \vartheta [A_a C] (a - b) \, \vartheta [A_a P_\beta C] (u - v) \, \vartheta [A_a P_\gamma C] (a + b)$$
$$\cdot \vartheta [A_a P_\beta P_\gamma C] (u + v)$$

$$= (C) \sum (P_a C) \begin{pmatrix} P_\beta P_\gamma \\ A_a \end{pmatrix} \vartheta [A_a] (a - v) \, \vartheta [A_a P_\beta] (u - b) \, \vartheta [A_a P_\gamma] (a + v)$$
$$\cdot \vartheta [A_a P_\beta P_\gamma] (u + b).$$

Making the arguments zero we have

$$\text{(5)} \qquad \sum (P_a C) \begin{pmatrix} P_\beta P_\gamma \\ A_a \ C \end{pmatrix} \vartheta [A_a C] \, \vartheta [A_a P_\beta C] \, \vartheta [A_a P_\gamma C] \, \vartheta [A_a P_\beta P_\gamma C]$$
$$= (C) \sum (P_a C) \begin{pmatrix} P_\beta P_\gamma \\ A_a \end{pmatrix} \vartheta [A_a] \, \vartheta [A_a P_\beta] \, \vartheta [A_a P_\gamma] \, \vartheta [A_a P_\beta P_\gamma].$$

This formula can be simplified by representing the characteristics of the GÖPEL system in the form:

$$AD_j, \quad A_\beta D_j, \quad A_\gamma D_j, \quad AA_\beta A_\gamma D_j \qquad (j = 0, 1, \cdots, \tfrac{1}{4}r - 1).$$

Equation (5) becomes:

$$\text{(6)} \qquad [1 + (P_\beta, C)][1 + (P_\gamma, C)] \sum (D_j C) \begin{pmatrix} D_j \\ AA_\beta A_\gamma \end{pmatrix} \vartheta [AD_j C]$$
$$\cdot \vartheta [A_\beta D_j C] \, \vartheta [A_\gamma D_j C] \, \vartheta [AA_\beta A_\gamma D_j C]$$
$$= [1 + (P_\beta, C)][1 + (P_\gamma, C)](C) \begin{pmatrix} C \\ A \end{pmatrix} \begin{pmatrix} A_\beta A_\gamma \\ C \end{pmatrix} \sum (D_j C) \begin{pmatrix} D_j \\ AA_\beta A_\gamma \end{pmatrix}$$
$$\cdot \vartheta [AD_j] \, \vartheta [A_\beta D_j] \, \vartheta [A_\gamma D_j] \, \vartheta [AA_\beta A_\gamma D_j].$$

If either (P_β, C) or (P_γ, C) is equal to -1, formula (6) vanishes identically. If $(P_\beta, C) = (P_\gamma, C) = 1$ we have

$$\text{(III)} \qquad \sum_j (D_j C) \begin{pmatrix} D_j \\ AA_\beta A_\gamma \end{pmatrix} \vartheta [AD_j C] \, \vartheta [A_\beta D_j C] \, \vartheta [A_\gamma D_j C] \, \vartheta [AA_\beta A_\gamma D_j C]$$
$$= (C) \begin{pmatrix} C \\ A \end{pmatrix} \begin{pmatrix} A_\beta A_\gamma \\ C \end{pmatrix} \sum_j (D_j C) \begin{pmatrix} D_j \\ AA_\beta A_\gamma \end{pmatrix} \vartheta [AD_j] \, \vartheta [A_\beta D_j] \, \vartheta [A_\gamma D_j]$$
$$\cdot \vartheta [AA_\beta A_\gamma D_j].$$

Further results are most easily obtained by starting with the formula:

$$(7) \quad \left[\sum_a \sqrt{P_a}\, \vartheta[A_a](u+v)(u-v) \right] \left[\sum_a \sqrt{P_a}\, \vartheta[A_a](a+b)(a-b) \right]$$
$$= \left[\sum_a \sqrt{P_a}\, \vartheta[A_a](u+b)(u-b) \right] \left[\sum \sqrt{P_a}\, \vartheta[A_a](a+v)(a-v) \right],$$

which is given in the memoir cited above (p. 200). We assume here that the characteristics A_a are all even. Replace u, b by $u+C$, $b+C$. We obtain:

$$\left[\sum \binom{C}{A_a} \sqrt{P_a}\, \vartheta[A_a C](u+v)(u-v) \right] \left[\sum \binom{C}{A_a} \sqrt{P_a}\, \vartheta[A_a C](b+a)(b-a) \right]$$
$$= \left[\sum (A_a, C)\sqrt{P_a}\, \vartheta[A_a](u+b)(u-b) \right] \left[\sum \sqrt{P_a}\, \vartheta[A_a](a+v)(a-v) \right].$$

Making the arguments zero and extracting the root we have

$$\sum \binom{C}{A_a} \sqrt{P_a}\, \vartheta^2[A_a C] = \sqrt{\sum (A_a, C)\sqrt{P_a}\, \vartheta^2[A_a] \cdot \sum \sqrt{P_a}\, \vartheta^2[A_a]}.$$

Multiply both members by $\sqrt{P_\beta}$, and add all the 2^p equations obtained by giving all possible combinations of signs to the radicals $\sqrt{P_a}$, $\sqrt{P_\beta}$. Using S to denote this summation and observing that $S\sqrt{P_a}\sqrt{P_\beta} = 2^p(P_a)$, or 0 according as a is or is not equal to β, we have

$$(IV) \quad 2^p \binom{C}{A_a}(P_a)\vartheta^2[A_a C] = S\sqrt{P_a}\sqrt{\sum (A_a, C)\sqrt{P_a}\, \vartheta^2[A_a] \cdot \sum \sqrt{P_a}\, \vartheta^2[A_a]}.$$

This formula vanishes identically if $(A_a C) = -1$.

The relation (IV) enables one to express all of the $2^{p-1}(2^p+1)$ theta constants homogeneously and algebraically in terms of 2^p of them, viz., the $\vartheta[A_a]$. Since the thetas depend on $\frac{1}{2}p(p+1)+1$ parameters (including the common multiplicative constant) it is evident that for $p > 2$ formula (IV) gives a parametric representation of the theta constants with $2^p - \frac{1}{2}p(p+1) - 1$ superfluous parameters; or, in other words, the 2^p parameters $\vartheta[A_a]$ satisfy this number of relations. These relations would seem to follow at once from formula (3), p. 204 of the memoir of FROBENIUS already cited when the arguments u are made to vanish.

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